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GEOMETRICALLY NONLINEAR ANALYSIS OF THE STRESS–STRAIN STATE OF TOROIDAL SHELLS UNDER PURE BENDING

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The stress–strain state of toroidal shells (curvilinear tubes) under bending with end moments was considered for the first time in [1, 2] where the problem was formulated and approximate solutions in the framework of small elastic displacements were found. In [3-6], on the basis of different approaches, refined solutions were obtained which make it possible to cover a wide range of variation of geometrical parameters of the tubes.

In [7], the nonlinear deformation of circular cylindrical shells under pure bending was studied, utilizing the assumptions [2] and the value of the limiting bending moment for which the loss of stability for the shell occurs was found. In [8, 9], attempts were made to improve the results [7] by means of retention of small terms, but as in [7] the linear relations for circumferential strain and the angle of rotation of the tangent to the cross-sectional contour were used. In [10], the problem of finite bending of cylindrical shell was reduced to a fourth-order system of two nonlinear differential equations which was solved in [11-14]. Nonlinear deformation and the stability of shells of elliptical cross section were treated in [15]. In [16], nonlinear equations for the bending of curvilinear tubes were derived, and an approximate analytical solution for the case of small initial curvature of the tube was obtained. Approximate solutions of the problem may also be found in [17, 18]. The formulation of the tube bending problem was discussed in [19, 20] from the standpoint of the geometrically nonlinear theory of shells. In [19], the nonlinear deformation of toroidal shells was also examined on the basis of numerical algorithms, and comparison with the results of other authors was carried out.

Analysis of the studies dealing with the Dubyaga–Karman–Brazier problem shows that in most of them shells of circular cross section were considered. The known analytical solutions are applicable only for shells with small initial curvature of the axis and do not make it possible to study the stress–strain state under finite bending which is accompanied by considerable distortion of the cross section. It is of interest to study the effect of geometrical nonlinearity on the magnitude and character of the stress distribution in shells as well as to estimate the range of applicability of the known approximate solutions on the basis of the refined approach.

We consider a thin-walled toroidal shell bent in the plane of curvature of its axial line with end moments M. Let the shape of the cross section (the meridian) be defined in the parametric form $x_i = x_i(s)$, where s is the arc length, and $i = 1, 2$. We assume that the cross sections which are normal to the axial line remain plane and normal to the axial line in the process of loading the shell, but can deform in their plane. The stress–strain state depends only on the coordinates, which agrees with the formulation of [1, 2, 7]. On the basis of the assumed assumptions and the Kirchhoff–Love hypotheses we write the equation of shell surface in its initial and deformed state in the vector form

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{e}_i x_i + z\mathbf{n},\ \mathbf{R}^\vee = \mathbf{R}_0^\vee + \mathbf{e}_i^\vee x_i^\vee + z\mathbf{n}^\vee\ (i = 1,2), \tag{1}$$

where $\mathbf{R}_0$ is the radius-vector of the axial line, $\mathbf{e}_i = \mathbf{e}_i(t)$ are the unit vectors lying in the plane of the cross section, t is the arc length of the axial line of the shell, $\mathbf{n} = \mathbf{e}_i\lambda_i^n$ is the unit normal vector to the middle surface of the shell, λ_i^n are the direction cosines of the normal vector, z is the normal coordinate to the middle surface of the shell, and the superscript $\vee$ denotes quantities which refer to the deformed state. Here and henceforth the rule of summation over repeating indexes is employed.

Using the Eq. (1), we obtain the relations for the strains and the curvature changes of the middle surface of the shell in the meridional and axial directions:

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TABLE 1

α	μ				
	0	1	2	5	10
	$\sigma_t^* r/Eh$				
0,5	0,141 (0,141)	0,119 (0,117)	0,087 (0,077)	0,057	0,047
1	0,262 (0,265)	0,198 (0,192)	0,145 (0,128)	0,105	0,088
1,5	0,341 (0,352)	0,235 (0,231)	0,187 (0,165)	0,146	0,126
2	0,367 (0,385)	0,260 (0,257)	0,221 (0,192)	0,184	0,161
2,5	0,359 (0,385)	0,283 (0,275)	0,253 (0,214)	0,219	0,192
3	0,360 (0,385)	0,307 (0,289)	0,282 (0,231)	0,250	0,223

$$\varepsilon_s = \frac{1}{2}(x_i^{\vee\prime} x_i^{\vee\prime} - 1),\ \varkappa_s = x_i^{\vee\prime} \lambda_i^{n\,\vee\prime} - x_i' \lambda_i^{n\prime},$$
$$\varepsilon_t = A_t^{-1}(\varepsilon + k^{\vee} x_1^{\vee} - k x_1),\ \varkappa_t = A_t^{-1}(k^{\vee} \lambda_1^{n\vee} - k\lambda_1^{n}). \tag{2}$$

Here, $A_t = 1 + kx_1$ is the Lamé parameter, ε and k are the strain and curvature of the axial line, and the prime denotes derivatives with respect to the coordinates.

The potential strain energy of a toroidal shell with unit length of the axial line has the form

$$\Pi = \frac{1}{2}\int (T_s \varepsilon_s + T_t \varepsilon_t + M_s \varkappa_s + M_t \varkappa_t) A_t ds, \tag{3}$$

where T_s, T_t, M_s, and M_t are the forces and the bending moments which for the case of an isotropic, linear-elastic body are connected with the strains and the curvatures changes (2) by the following relations:

$$\begin{gathered} T_s = B(\varepsilon_s + \nu\varepsilon_t),\ T_t = B(\varepsilon_t + \nu\varepsilon_s), \\ M_s = D(\varkappa_s + \nu\varkappa_t),\ M_t = D(\varkappa_t + \nu\varkappa_s), \\ B = Eh(1 - \nu^2)^{-1},\ D = Bh^2/12. \end{gathered} \tag{4}$$

Here, E is Young's modulus, ν is Poisson's ratio, and h is the thickness of the shell.

The total potential energy has the form $U = \Pi - A$, where $A = M(k^{\vee} - k)$ is the work of the external bending moments.

We divide the shell into finite elements with length l in the meridional direction. Writing the Taylor series expansion of the unknown functions and neglecting small terms of order $O(l^2)$, we obtain an approximate variant of the deformation relations (2) for a finite element:

$$\begin{aligned} &\varepsilon_t = A_t^{-1}(\varepsilon + k^{\vee} x_1^{\vee} - k x_1),\ \varkappa_t = A_t^{-1}(k^{\vee} \lambda_1^{n\vee} - k\lambda_1^{n}), \\ &\varepsilon_s = \frac{1}{2}(b_i b_k x_{ji}^{\vee} x_{jk}^{\vee} - 1), \\ &\varkappa_s = N_i \theta_i,\ \theta_i = b_k(\lambda_{ji}^{n\,\vee} x_{jk}^{\vee} - \lambda_{ji}^{n} x_{jk}), \\ &A_t = 1 + kx_1,\ x_1 = \frac{1}{2}(x_{11} + x_{12}),\ \lambda_1^n = \frac{1}{2}(\lambda_{11}^n + \lambda_{12}^n), \\ &b_1 = -b_2 = -1/l,\ N_1 = (6s - 4l)l^{-2},\ N_2 = (6s - 2l)l^{-2} \end{aligned} \tag{5}$$

(x_{ij}, λ_{ij}^n(i, j = 1, 2) are the coordinates and direction cosines of the normal vector at the j-th node of an element). We introduce the five-components vector of the generalized elastic displacements

$$u^T = |\varepsilon_s, \theta_1, \theta_2, \varepsilon_t, \varkappa_t|.$$

TABLE 2

α	μ = 0	1	2	5	10
	$\sigma_s^* r/Eh$				
0,5	0,032 (0,031)	0,088 (0,094)	0,121 (0,156)	0,127	0,103
1	0,133 (0,125)	0,234 (0,250)	0,273 (0,375)	0,254	0,205
1,5	0,309 (0,281)	0,413 (0,469)	0,433 (0,656)	0,377	0,306
2	0,531 (0,500)	0,594 (0,750)	0,584 (1,000)	0,494	0,404
2,5	0,748 (0,781)	0,758 (1,094)	0,723 (1,406)	0,605	0,501
3	0,935 (1,125)	0,903 (1,500)	0,848 (1,875)	0,712	0,594

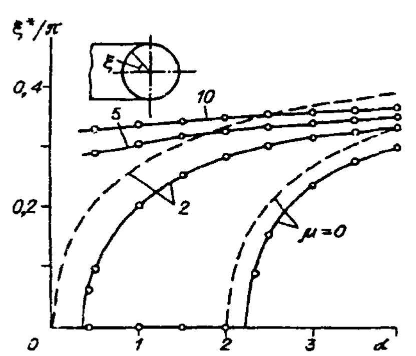

Fig. 1

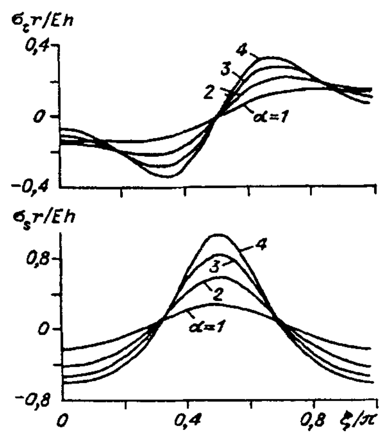

Fig. 2

Substituting the relations (4) and (5) into (3) and integrating between the limits 0 and l, we obtain the potential strain energy of an element in the form $\Pi = (1/2)\mathbf{u}^T\mathbf{K}\mathbf{u}$, where the nonzero coefficients of the symmetric stiffness matrix $\mathbf{K}$ are given by the expressions

$$
\begin{aligned}
&K_{11} = BA_t l,\ K_{14} = \nu K_{11},\ K_{22} = Dl^{-1}[4 + k(3x_{11} + x_{12})],\\
&K_{23} = 2DA_t l^{-1},\ K_{25} = -\nu D(1 + kx_{11}),\\
&K_{33} = Dl^{-1}[4 + k(x_{11} + 3x_{12})],\ K_{35} = \nu D(1 + kx_{12}),\\
&K_{44} = K_{11},\ K_{55} = DA_t l.
\end{aligned}
$$

One feature of the given formulation of the problem is that a finite element of the shell contains the following nodal and non-nodal unknowns which form the vector of the generalized coordinates $\mathbf{q}$:

$$\mathbf{q}^T = |x_{11}^v, x_{21}^v, \varphi_1^v, x_{12}^v, x_{22}^v, \varphi_2^v, \varepsilon, k^v|$$

(φ_i^V is the angle of rotation of the normal vector at the i-th node).

We write the Taylor series expansion of the potential energy of an element with respect to the increments of the generalized coordinates $\delta\mathbf{q}$ in the neighborhood of a certain deformed state:

$$\Pi = \Pi_0 + \delta\Pi + \frac{1}{2}\delta^2\Pi + \ldots, \qquad (6)$$

$$\delta\Pi = \mathbf{g}^T\delta\mathbf{q},\ \delta^2\Pi = \delta\mathbf{q}^T\mathbf{H}\delta\mathbf{q}$$

TABLE 3

α	m	α	m
−4,8	−1,00	14,4	0,586
−2,4	−0,395	16,8	0,594
2,4	0,245	19,2	0,595
4,8	0,393	24	0,586
9,6	0,537		

(g and **H** are the gradient and the Hess matrix of the potential energy). When the potential energy is a quadratic function of the components of the vector **u**, one obtains the following expressions [21]:

$$\mathbf{g} = \mathbf{u}'\mathbf{P},\ \mathbf{P} = \mathbf{K}\mathbf{u},$$
$$\mathbf{H} = \mathbf{u}'\mathbf{K}\mathbf{u}'^{T} + P_i\mathbf{u}''_i\ (i = 1,...,5)$$

($\mathbf{u}'$ and $\mathbf{u}''_i$ are the matrices containing the first and second derivatives of the components of the vector **u** with respect to the generalized coordinates).

If we confine ourselves to the given terms of the expansion (6), use of the principle of stationary potential energy $\delta U = 0$ for the assemblage of finite elements will lead to the system of equations

$$\mathbf{H}\delta\mathbf{q} + \mathbf{g} - \mathbf{Q} = 0, \tag{7}$$

where **g** and **H** are the gradient and the Hess matrix of the assemblage of finite elements, and **Q** is the vector of the generalized external forces. Solving the system of equations (7) represents one step in the iterative process of finding the equilibrium state of the discrete system. After the quantities $\delta\mathbf{q}$ are determined, the new values of the unknowns are calculated by means of the following (no summation over j):

$$(x_{ij}^{\vee})^* = x_{ij}^{\vee} + \delta x_{ij}^{\vee},\ (\lambda_{ij}^{n\vee})^* = \lambda_{ij}^{n\vee}\cos\delta\varphi_j^{\vee} + \lambda_{ij}^{\vee}\sin\delta\varphi_j^{\vee},$$
$$\varepsilon^* = \varepsilon + \delta\varepsilon,\ (k^{\vee})^* = k^{\vee} + \delta k^{\vee}$$

($\lambda_{ij}^{\vee}$ are the direction cosines of the unit vector, normal to $\mathbf{n}_j^{\vee}$). The process of solution in accordance with scheme (7) is repeated until the specified accuracy of determining the unknowns is satisfied.

To calculate the toroidal shells, one needs as initial data the values of the coordinates and the direction cosines of the normal vector at the nodes of the cross section under consideration.

Let us investigate the stress state of toroidal shells of circular cross section with radius r, which are characterized by various values of curvature parameter $\mu = (12(1 - \nu^2))^{1/2}kr^2/h$ with $r/h = 100$ and $\nu = 0$. In Tables 1 and 2 we give the dependences of the maximum values of axial stresses σ_t^* and circumferential (meridional) stresses σ_s^* on the parameter of curvature change or the axial line $\alpha = \mu^{\vee} - \mu$.

It should be noted that the point having the coordinate ξ^*, where the maximum axial stresses σ_t^* occur, is displaced in the process of deformation of the shell toward the neutral line (Fig. 1). Moreover, this displacement is most noticeable for shells with small values of the curvature parameter ($\mu < 5$). When $\mu > 10$ the maximum stresses occur in the neighborhood of the neutral line and the corresponding coordinate ξ^* varies slightly with bending of the shell. The approximate solutions [7, 16] represented by the dashed curves in Fig. 1 and the bracketed values in the Tables 1 and 2 describe satisfactorily the stress state only for shells with small initial curvature ($\mu \lesssim 1$). The results of [16] lead to the largest error in determining the maximum circumferential stresses σ_s^* which occur at the point $\xi = \pi/2$. Thus, for the case of $\mu = 1$ in the region of small curvature changes of the shell axis ($\alpha < 0.5$) the relative error amounts to 6% and increases to 26% for $\alpha = 2$.

In Fig. 2 we show the distribution of stresses over the section of the shell with parameter $\mu = 2$ for various values of α. As a result of flattening the cross section the circumferential stresses σ_s, which occur mainly due to bending of the wall, have the largest magnitude.

Let us consider bending of a toroidal shell of noncircular section whose form is described by the expression [22]

$$\theta = \xi + 0{,}5\sin\xi - 0{,}9452\sin 2\xi + 0{,}3\sin 3\xi - 0{,}4\sin 4\xi,\ \xi = s/r,$$

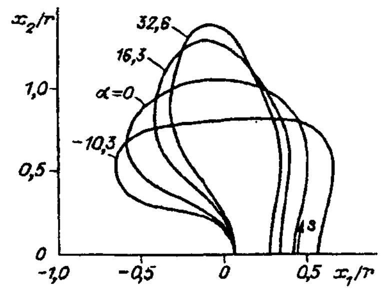

Fig. 3

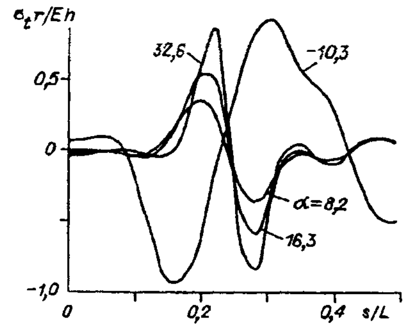

Fig. 4

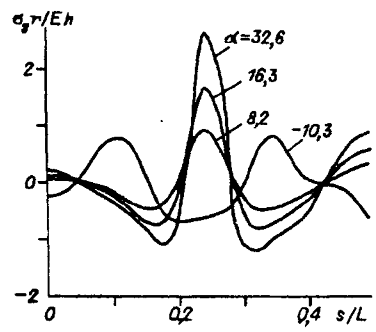

Fig. 5

where θ is the angle between the normal vector **n** and the x_1 axis, $r = L/2\pi$ is the reduced radius of the section, and L is the perimeter of the section. The Cartesian coordinates are determined by numerical integration of the relations $x_1' = -\sin\theta$ and $x_2' = \cos\theta$.

Analysis of the linear problem of bending of shells with various curvature parameters of the axial line indicates that in the region of large values of $\mu > 30$ for the flexibility factor f the formula $f = 0.155\mu$ is valid.

In Table 3 we represent the dependence between the dimensionless parameters of bending moment $m = (12(1 - \nu^2))^{1/2}Mr^2/hEI$ (I being the moment of inertia of the cross section) and the curvature change of the axial line α of the shell for the case $r/h = 100$, $\nu = 0.3$, and $\mu = 50$. The linear solution of the problem $m = \alpha/f$ is acceptable in the narrow range $-0.4 \lesssim \alpha \lesssim 0.4$. The value of the limiting moment corresponding to the bending of the shell ($\alpha > 0$) amounts to $m \approx 0.6$.

The forms of deformed cross section for various values of α are presented in Fig. 3. In Figs. 4 and 5 we show the distribution of the stresses over the section of the shell. The axial stresses were calculated for the middle surface, while the meridional ones were calculated for the outer surface of the shell. It should be noted that for $\alpha > 0$ the stresses are localized in the neighborhood of the neutral line, the meridional stresses exceeding the axial stresses by a factor 2-2.5.

We remark finally that it took about 4 min to calculate the stress–strain state of the shell for the range $-4.8 < \alpha < 24$ with double precision calculations carried out on a PC AT 286 computer.

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